11-4 Geometric Series

Main Ideas

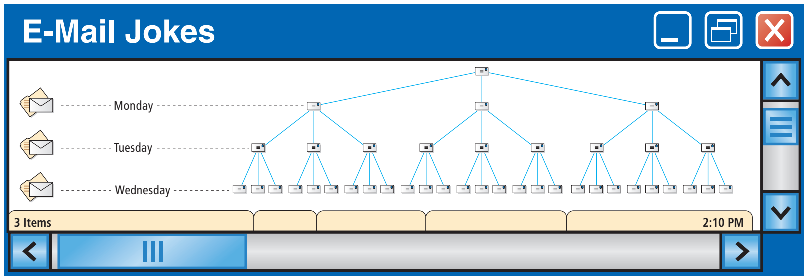
- Find sums of geometric series.
- Find specific terms of geometric series.

New Vocabulary

geometric series

GET READY for the Lesson

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



Geometric Series Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. The indicated sum of the numbers in the sequence, $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, is called a **geometric series**.

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above. Multiply each term in the series by the common ratio and subtract the result from the original series.

$$\begin{array}{rl} S_8 = & 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\ (-)\ 3S_8 = & \phantom{1 +{}} 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\ \hline (1-3)S_8 = & 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561 \end{array}$$

first term in series

last term in series multiplied by common ratio; in this case, a_9

$$S_8 = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$

common ratio

The expression for S_8 can be written as $S_8 = \dfrac{a_1 - a_1r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Study Tip

Terms of Geometric Sequences

Remember that a_9 can also be written as a_1r^8.

KEY CONCEPT — Sum of a Geometric Series

The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \text{ or } S_n = \frac{a_1(1 - r^n)}{1 - r}, \text{ where } r \neq 1.$$

Real-World Link The development of vaccines for many diseases has helped to prevent infection. Vaccinations are commonly given to children.

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \cdots + 4$ is such a series. In general, the sum of n terms of a geometric series with $r = 1$ is $n \cdot a_1$.

Real-World EXAMPLE — Find the Sum of the First *n* Terms

1 **HEALTH Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic, and each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?**

This is a geometric series with $a_1 = 5$, $r = 4$, and $n = 10$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Sum formula}$$

$$S_{15} = \frac{5(1 - 4^{10})}{1 - 4} \quad n = 10, a_1 = 5, r = 4$$

$$S_{15} = 1{,}747{,}625 \quad \text{Use a calculator.}$$

After ten weeks, 1,747,625 people have been affected by the illness.

CHECK Your Progress

1. **GAMES** Maria arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Maria use?

You can use sigma notation to represent geometric series.

EXAMPLE — Evaluate a Sum Written in Sigma Notation

2 **Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.**

Method 1 Since the sum is a geometric series, you can use the formula.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_6 = \frac{5(1 - 2^6)}{1 - 2} \quad n = 6, a_1 = 5, r = 2$$

$$S_6 = \frac{5(-63)}{-1} \quad 2^6 = 64$$

$$S_6 = 315 \quad \text{Simplify.}$$

Method 2 Find the terms by replacing n with 1, 2, 3, 4, 5, and 6. Then add.

$$\sum_{n=1}^{6} 5 \cdot 2^{n-1}$$

$= 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{3-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1})$ Write out the sum.

$= 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32)$ Simplify.

$= 5 + 10 + 20 + 40 + 80 + 160$ Multiply.

$= 315$ Add.

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{5} -4 \cdot 3^{n-1}$.

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How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? You can use the formula for the nth term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$, to find an expression involving r^n.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r$ Multiply each side by r.

$a_n \cdot r = a_1 \cdot r^n$ $r^{n-1} \cdot r^1 = r^{n-1+1}$ or r^n

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

EXAMPLE 3 Use the Alternate Formula for a Sum

Find the sum of a geometric series for which $a_1 = 15{,}625$, $a_n = -5$, and $r = -\frac{1}{5}$.

Since you do not know the value of n, use the formula derived above.

$S_n = \frac{a_1 - a_n r}{1 - r}$ Alternate sum formula

$= \frac{15{,}625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}$ $a_1 = 15{,}625$; $a_n = -5$; $r = -\frac{1}{5}$

$= \frac{15{,}624}{\frac{6}{5}}$ or 13,020 Simplify.

CHECK Your Progress

3. Find the sum of a geometric series for which $a_1 = 1000$, $a_n = 125$, and $r = \frac{1}{2}$.

Specific Terms You can use the formula for the sum of a geometric series to help find a particular term of the series.

EXAMPLE 4 Find the First Term of a Series

Find a_1 in a geometric series for which $S_8 = 39{,}360$ and $r = 3$.

$S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ Sum formula

$39{,}360 = \dfrac{a_1(1 - 3^8)}{1 - 3}$ $S_8 = 39{,}360$; $r = 3$; $n = 8$

$39{,}360 = \dfrac{-6560a_1}{-2}$ Subtract.

$39{,}360 = 3280a_1$ Divide.

$12 = a_1$ Divide each side by 3280.

CHECK Your Progress

4. Find a_1 in a geometric series for which $S_7 = 258$ and $r = -2$.

CHECK Your Understanding

Example 1 (p. 644)

Find S_n for each geometric series described.

1. $a_1 = 5, r = 2, n = 14$
2. $a_1 = 243, r = -\frac{2}{3}, n = 5$

Find the sum of each geometric series.

3. $54 + 36 + 24 + 16 + \cdots$ to 6 terms
4. $3 - 6 + 12 - \cdots$ to 7 terms

5. **WEATHER** Heavy rain caused a river to rise. The river rose three inches the first day, and each day it rose twice as much as the previous day. How much did the river rise in five days?

Example 2 (pp. 644–645)

Find the sum of each geometric series.

6. $\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$
7. $\sum_{n=1}^{7} 81\left(\frac{1}{3}\right)^{n-1}$
8. $\sum_{n=1}^{12} \frac{1}{6}(-2)^n$
9. $\sum_{n=1}^{8} \frac{1}{3} \cdot 5^{n-1}$
10. $\sum_{n=1}^{6} 100\left(\frac{1}{2}\right)^{n-1}$
11. $\sum_{n=1}^{9} \frac{1}{27}(-3)^{n-1}$

Example 3 (p. 645)

Find S_n for each geometric series described.

12. $a_1 = 12, a_5 = 972, r = -3$
13. $a_1 = 3, a_n = 46{,}875, r = -5$
14. $a_1 = 5, a_n = 81{,}920, r = 4$
15. $a_1 = -8, a_6 = -256, r = 2$

Example 4 (p. 646)

Find the indicated term for each geometric series described.

16. $S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1$
17. $S_n = 33, a_n = 48, r = -2; a_1$
18. $S_n = 443, r = \frac{1}{3}, n = 6; a_1$
19. $S_n = -242, a_n = -162, r = 3; a_1$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
20–25, 28–31	1
26, 27	3
32, 33	2
34–37	4

Find S_n for each geometric series described.

20. $a_1 = 2, a_6 = 486, r = 3$

21. $a_1 = 3, a_8 = 384, r = 2$

22. $a_1 = 4, r = -3, n = 5$

23. $a_1 = 5, r = 3, n = 12$

24. $a_1 = 2401, r = -\frac{1}{7}, n = 5$

25. $a_1 = 625, r = \frac{3}{5}, n = 5$

26. $a_1 = 1296, a_n = 1, r = -\frac{1}{6}$

27. $a_1 = 343, a_n = -1, r = -\frac{1}{7}$

28. GENEALOGY In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

29. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have \$1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

Day	Payment
1	1¢
2	2¢
3	4¢
4	8¢
⋮	⋮
30	?
Total	?

Real-World Link

Some of the best-known legends involving a king are the Arthurian legends. According to the legends, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find the sum of each geometric series.

30. $4096 - 512 + 64 - \cdots$ to 5 terms

31. $7 + 21 + 63 + \cdots$ to 10 terms

32. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$

33. $\sum_{n=1}^{6} 2(-3)^{n-1}$

Find the indicated term for each geometric series described.

34. $S_n = 165, a_n = 48, r = -\frac{2}{3}; a_1$

35. $S_n = 688, a_n = 16, r = -\frac{1}{2}; a_1$

36. $S_n = -364, r = -3, n = 6; a_1$

37. $S_n = 1530, r = 2, n = 8; a_1$

Find S_n for each geometric series described.

38. $a_1 = 162, r = \frac{1}{3}, n = 6$

39. $a_1 = 80, r = -\frac{1}{2}, n = 7$

40. $a_1 = 625, r = 0.4, n = 8$

41. $a_1 = 4, r = 0.5, n = 8$

42. $a_2 = -36, a_5 = 972, n = 7$

43. $a_3 = -36, a_6 = -972, n = 10$

44. $a_1 = 4, a_n = 236{,}196, r = 3$

45. $a_1 = 125, a_n = \frac{1}{125}, r = \frac{1}{5}$

Find the sum of each geometric series.

46. $\frac{1}{16} + \frac{1}{4} + 1 + \cdots$ to 7 terms

47. $\frac{1}{9} - \frac{1}{3} + 1 - \cdots$ to 6 terms

48. $\sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^{n-1}$

49. $\sum_{n=1}^{20} 3 \cdot 2^{n-1}$

50. $\sum_{n=1}^{16} 4 \cdot 3^{n-1}$

51. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

EXTRA PRACTICE
See pages 915, 936.

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Find the indicated term for each geometric series described.

52. $S_n = 315, r = 0.5, n = 6; a_2$

53. $S_n = 249.92, r = 0.2, n = 5, a_3$

54. WATER TREATMENT A certain water filtration system can remove 80% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system three times, what percent of the original contaminants will be removed from the water sample?

Graphing Calculator

Use a graphing calculator to find the sum of each geometric series.

55. $\sum_{n=1}^{20} 3(-2)^{n-1}$

56. $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$

57. $\sum_{n=1}^{10} 5(0.2)^{n-1}$

58. $\sum_{n=1}^{13} 6\left(\frac{1}{3}\right)^{n-1}$

H.O.T. Problems

59. OPEN ENDED Write a geometric series for which $r = \frac{1}{2}$ and $n = 4$.

60. REASONING Explain how to write the series $2 + 12 + 72 + 432 + 2592$ using sigma notation.

61. CHALLENGE If a_1 and r are integers, explain why the value of $\frac{a_1 - a_1r^n}{1 - r}$ must also be an integer.

62. REASONING Explain, using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 - 1}{x - 1}$, assuming $x \neq 1$.

63. *Writing in Math* Use the information on page 643 to explain how e-mailing a joke is related to a geometric series. Include an explanation of how the situation could be changed to make it better to use a formula than to add terms.

STANDARDIZED TEST PRACTICE

64. ACT/SAT The first term of a geometric series is −1, and the common ratio is −3. How many terms are in the series if its sum is 182?

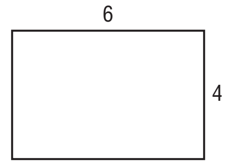
A 6

B 7

C 8

D 9

65. REVIEW Which set of dimensions corresponds to a rectangle similar to the one shown below?



F 3 units by 1 unit

G 12 units by 9 units

H 13 units by 8 units

J 18 units by 12 units

Spiral Review

Find the geometric means in each sequence. (Lesson 11-3)

66. $\frac{1}{24}$, __?__, __?__, __?__, 54

67. -2, __?__, __?__, __?__, __?__, $-\frac{243}{16}$

Find the sum of each arithmetic series. (Lesson 11-2)

68. $50 + 44 + 38 + \cdots + 8$

69. $\sum_{n=1}^{12} (2n + 3)$

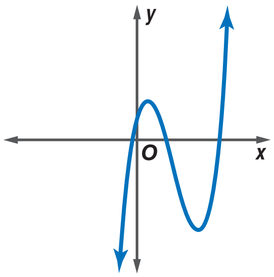
Solve each equation. Check your solutions. (Lesson 8-6)

70. $\frac{1}{y+1} - \frac{3}{y-3} = 2$

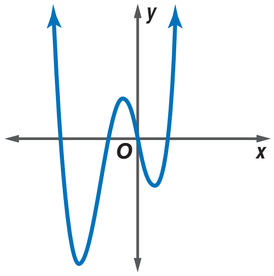
71. $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$

Determine whether each graph represents an odd-degree polynomial function or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-4)

72.



73.



Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 5-3)

74. $3d^2 + 2d - 8$

75. $42pq - 35p + 18q - 15$

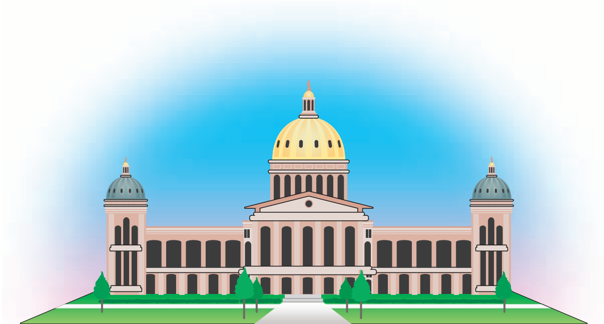
76. $13xyz + 3x^2z + 4k$

VOTING **For Exercises 77–79, use the table that shows the percent of the Iowa population of voting age that voted in each presidential election from 1984–2004.** (Lesson 2-5)

77. Draw a scatter plot in which x is the number of elections since the 1984 election.

78. Find a linear prediction equation.

79. Predict the percent of the Iowa voting age population that will vote in the 2012 election.



Year	1984	1988	1992	1996	2000	2004
Percent	76.0	75.7	72.4	80.2	73.2	75.0

Source: sos.state.ia.us

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $\frac{a}{1-b}$ for the given values of a and b.** (Lesson 1-1)

80. $a = 1, b = \frac{1}{2}$

81. $a = 3, b = -\frac{1}{2}$

82. $a = \frac{1}{3}, b = -\frac{1}{3}$

83. $a = \frac{1}{2}, b = \frac{1}{4}$

84. $a = -1, b = 0.5$

85. $a = 0.9, b = -0.5$